Feynman's path integral and mutually unbiased bases

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Abstract
Our previous work on quantum mechanics in Hilbert spaces of finite dimension N is applied to elucidate the deep meaning of Feynman's path integral pointed out by G Svetlichny. He speculated that the secret of the Feynman path integral may lie in the property of mutual unbiasedness of temporally proximal bases. We confirm the corresponding property of the short-time propagator by using a specially devised $N \times N$ approximation of quantum mechanics in $L^2(\mathbb{R})$ applied to our finite-dimensional analogue of a free quantum particle.

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1. Introduction

The circle of ideas which generated Feynman path integrals is contained in works by Dirac and Feynman [1, 2]. Especially in the latter work the representation of quantum mechanical evolution amplitudes in terms of heuristic integrals on 'path space' was developed into an alternative general formulation of quantum dynamics, equivalent to the previous formulations by Heisenberg, Schrödinger or Schwinger.

The miraculous success of Feynman's method in dealing with quantum fields should be, however, mitigated by the fact that there is so far no rigorous way to define it in terms of conventional measure theory. Namely, the heuristic expression for Feynman path integrals is in terms of a complex formal density which does not define a measure. Thus the mathematical definition of the objects understood under the name of Feynman path integrals posed genuine new problems which have been attacked by different methods. For instance, Feynman's path integral in non-relativistic quantum mechanics is conventionally viewed as a formal expression which can be given meaning by a specially devised limiting process.

The diverse aspects and approaches make clear that the subject of Feynman path integrals should not be considered as a closed one, on the contrary, much work is needed on the conceptual mathematical and physical level in order to bring to fruition all the beautiful potentialities contained in those ideas. In this direction it was recently speculated by Svetlichny

[3, 4] that the secret of the path integral may rest on the mutual unbiasedness of temporally proximal bases. He focused the essential problems into the following questions:

(i) For what unitary groups $U(t)$ in $L^2(\mathbf{R}^n)$ do the position bases at times 0 and t tend to mutual unbiasedness as $t \to 0$?

(ii) Is there a discrete version of the previous question in a finite-dimensional Hilbert space which approximately simulates the propagation of a free particle?

(iii) What is the information-theoretic nature of the normalization factor A in the short-time propagator?

In this paper we elucidate the true meaning of these questions in the case of a *non-relativistic quantum particle on the real line*.

In section 2 we recall related notions of complementary observables and of mutually unbiased bases. Then the basic concepts of quantum mechanics in finite-dimensional Hilbert spaces are introduced in section 3. Section 4 is devoted to the construction of finite quantum phase space from the finite Heisenberg group. The group of inner automorphisms of finite quantum phase space is described in section 5. After these prerequisites, the $N \times N$ approximation of quantum mechanics on the real line is constructed in section 6 and applied to an analogue of a free non-relativistic particle. This approximation, being used in the Feynman short-time propagator (section 7) leads to the emergence of a Lagrangian as the corresponding local phase and at the same time demonstrates mutual unbiasedness of temporally proximal bases (section 8). Our derivation also yields the normalization factor A as a direct counterpart of the constant $1/\sqrt{N}$ involved in the definition of mutual unbiasedness.

2. Complementarity and mutually unbiased bases

Mutually unbiased bases in Hilbert spaces of finite dimensions are closely related to the quantal notion of complementarity. Namely, two observables A and B of a quantum system with Hilbert space of finite dimension N are called *complementary* [5], if their eigenvalues are non-degenerate and any two normalized eigenvectors $|u_i\rangle$ of A and $|v_j\rangle$ of B satisfy

$$|\langle u_i|v_j\rangle| = \frac{1}{\sqrt{N}}.$$

Then in an eigenstate $|u_i\rangle$ of A all eigenvalues $b_1, \ldots, b_N$ of B are measured with equal probabilities, and vice versa. This means that exact knowledge of the measured value of A implies *maximal uncertainty* to any measured value of B.

According to Wootters [6], two orthonormal bases in an N-dimensional complex Hilbert space

$$\{|u_i\rangle | i = 1, 2, \ldots, N\} \quad \text{and} \quad \{|v_j\rangle | j = 1, 2, \ldots, N\}$$

are called *mutually unbiased*, if inner products between all possible pairs of vectors taken from distinct bases have the same magnitude $1/\sqrt{N}$,

$$|\langle u_i|v_j\rangle| = \frac{1}{\sqrt{N}} \quad \text{for all} \quad i, j \in \{1, 2, \ldots, N\}.$$

Thus if the system is in the state $|u_i\rangle$, then transitions to any of the states $|v_j\rangle$ have equal probabilities.

It is important to note that, in an N-dimensional Hilbert space, there cannot be more than $N + 1$ mutually unbiased bases. It has also been proved that the maximal number of $N + 1$ mutually unbiased bases is attained, if N is a power of a prime [7].

3. Quantum mechanics in finite-dimensional Hilbert spaces

The mathematical arena for ordinary quantum mechanics is, due to Heisenberg's commutation relations, the infinite-dimensional Hilbert space. A useful model for quantum mechanics in a Hilbert space of finite dimension N is due to Weyl [8]. Its geometric interpretation as the simplest quantum kinematic on a finite discrete configuration space formed by a periodic chain of N points was elaborated by Schwinger [9]. In [10] we proposed a group theoretical formulation of this quantum model as well as a finite-dimensional analogue of the quantum evolution operator for a free particle.

In an N-dimensional Hilbert space with orthonormal basis $\mathcal{B} = \{|0\rangle, |1\rangle, \ldots |N-1\rangle\}$ the Weyl pair of unitary operators (Q_N, P_N) is defined by the relations

$$Q_N|\rho\rangle = \omega_N^\rho|\rho\rangle, \quad \rho = 0, 1, \ldots, N-1,$$

$$P_N|\rho\rangle = |\rho - 1 \pmod{N}\rangle,$$

where $\omega_N = \exp(2\pi i/N)$ [8] (see also [11, 12]). If $\mathcal{B}$ is the canonical basis of $\mathbb{C}^N$, the operators Q_N and P_N are represented by the matrices

$$Q_N = \mathrm{diag}\left(1, \omega_N, \omega_N^2, \ldots, \omega_N^{N-1}\right)$$

and

$$P_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

They fulfil a commutation relation

$$P_N Q_N = \omega_N Q_N P_N, \tag{1}$$

which is analogous to the relation for Weyl's exponential form of Heisenberg's commutation relations. Further, $P_N^N = Q_N^N = I_N, \omega_N^N = 1$.

The *finite Heisenberg group* is generated by ω_N, Q_N and P_N:

$$\Pi_N = \left\{\omega_N^l Q_N^j P_N^\sigma \,\middle|\, l, j, \sigma = 0, 1, 2, \ldots, N-1\right\}.$$

It consists of N^3 unitary $N \times N$ matrices and is also called the Pauli group.

The *geometrical picture* behind the above operators is the following [10]. The cyclic group $\mathbb{Z}_N = \{0, 1, \ldots N-1\}$ is the configuration space for N-dimensional quantum mechanics. Elements of this periodic chain $\mathbb{Z}_N$ provide labels of the vectors of the basis $\mathcal{B} = \{|0\rangle, |1\rangle, \ldots |N-1\rangle\}$ with the physical interpretation that $|\rho\rangle$ is the (normalized) eigenvector of position at $\rho \in \mathbb{Z}_N$. The action of $\mathbb{Z}_N$ on $\mathbb{Z}_N$ via addition modulo N is represented by unitary operators $U(\sigma) = P_N^\sigma$. The action of these discrete translations on vectors $|\rho\rangle$ from basis $\mathcal{B}$ is given by

$$U(\sigma)|\rho\rangle = P_N^\sigma|\rho\rangle = |\rho - \sigma \pmod{N}\rangle.$$

The important *discrete Fourier transformation* is given by the unitary Sylvester matrix S_N with elements

$$(S_N)_{k\rho} = \langle \rho|k\rangle = \frac{\omega_N^{\rho k}}{\sqrt{N}}.$$

The operator relation

$$S_N^{-1} P_N S_N = Q_N$$

shows that the discrete Fourier transform diagonalizes the momentum operator. In other words, it performs the transition from the coordinate representation to the momentum representation:

$$|k\rangle = \sum_{\rho=0}^{N-1} |\rho\rangle\langle\rho|k\rangle.$$

4. Finite quantum phase space

The following developments will heavily use the *finite phase space* Γ_N which is simply related to the finite Heisenberg group [13]. The centre $Z(\Pi_N)$ of the finite Heisenberg group is the set of all those elements of Π_N which commute with all elements in Π_N

$$Z(\Pi_N) = \{(l, 0, 0) | l = 0, 1, \ldots, N-1\}.$$

Since the centre is a normal subgroup, one can go over to the quotient group $\Pi_N/Z(\Pi_N)$. Its elements are the cosets labelled by pairs (j, σ), $j, \sigma = 0, 1, \ldots, N-1$. The quotient group can be identified with the finite phase space

$$\Gamma_N = \mathbb{Z}_N \times \mathbb{Z}_N, \qquad N = 2, 3, \ldots.$$

To simplify the notation, we denote the cosets corresponding to elements (j, σ) of the phase space Γ_N by $Q^j P^\sigma$ without subscripts N:

$$Q^j P^\sigma = \left\{\omega_N^l Q_N^j P_N^\sigma | l = 0, 1, \ldots, N-1\right\}.$$

The correspondence

$$\phi : \Pi_N/Z(\Pi_N) \to \Gamma_N = \mathbb{Z}_N \times \mathbb{Z}_N : Q^j P^\sigma \mapsto (j, \sigma)$$

is an isomorphism of Abelian groups, since

$$\phi((Q^j P^\sigma)(Q^{j'} P^{\sigma'})) = \phi((Q^j P^\sigma))\phi((Q^{j'} P^{\sigma'})) = (j, \sigma) + (j', \sigma') = (j + j', \sigma + \sigma').$$

The group of automorphisms of the quantum phase space Γ_N was studied in [13, 14]. The latter paper considered—instead of cosets—the one-dimensional grading subspaces of the Pauli-graded Lie algebra $gl(N, \mathbb{C})$ and studied their transformations under the automorphisms of $gl(N, \mathbb{C})$. The subgroup of *inner automorphisms* was induced by the action

$$\psi_X(A) = X^{-1}AX$$

of matrices X from $GL(N, \mathbb{C})$.

Along the same lines we consider those automorphisms of the above form, acting on elements of Π_N, which induce permutations of cosets in $\Pi_N/Z(\Pi_N)$. Operators X which induce these automorphisms are unitary. Explicit forms of these operators are given in [13] for N prime and in [14] for arbitrary N but only for special transformations of Γ_N.

Automorphisms ψ of the given form are equivalent if they define the same transformation of cosets in $\Pi_N/Z(\Pi_N)$:

$$\psi_Y \sim \psi_X \qquad \Leftrightarrow \qquad Y^{-1} Q^j P^\sigma Y = X^{-1} Q^j P^\sigma X$$

for all $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$. The group $\Pi_N/Z(\Pi_N)$ has two generators, the cosets P and Q. Hence if ψ_Y induces a transformation of $\Pi_N/Z(\Pi_N)$, then there must exist elements $a, b, c, d \in \mathbb{Z}_N$ such that

$$Y^{-1} Q Y = Q^a P^b \qquad \text{and} \qquad Y^{-1} P Y = Q^c P^d.$$

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It follows that to each equivalence class of automorphisms ψ_Y a quadruple (a, b, c, d) of elements in $\mathbb{Z}_N$ is assigned. Then we have the following theorem.

Theorem 1 [14]. *There is an isomorphism Φ between the set of equivalence classes of inner automorphisms ψ_Y and the group $SL(2, \mathbb{Z}_N)$ of 2×2 matrices with determinant equal to 1 modulo N,*

$$\Phi(\psi_Y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}_N;$$

the action of these automorphisms on $\Pi_N/Z(\Pi_N)$ is given by the right action of $SL(2, \mathbb{Z}_N)$ on the phase space $\Gamma_N = \mathbb{Z}_N \times \mathbb{Z}_N$,

$$(j', \sigma') = (j, \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

5. $N \times N$ approximation of quantum mechanics on the real line

An interesting approximation method in quantum mechanics was proposed in Husstad's Dr.ing. thesis [15] supervised by T Digernes at NTNU Trondheim. Their approach was inspired by an idea of Schwinger [16].

They approximate quantum operators in $L^2(\mathbb{R})$ for one-dimensional quantum systems by $N \times N$ matrices—operators in the Hilbert space $l^2(Z_N)$ of finite-dimensional quantum mechanics. To this end an auxiliary factor

$$\eta_N = \sqrt{\frac{2\pi}{N}}$$

is introduced. We have found that for our purpose of approximating the Feynman path integral it is still necessary to introduce two additional *dimensional quantities*: the *length unit a* and the corresponding *unit of linear momentum $\hbar/a$*. Then the position operator is approximated by the multiplication operator in position representation

$$q_N|\rho\rangle = a\eta_N\rho|\rho\rangle.$$

For the momentum operator p_N Schwinger had the real insight to define it as the discrete Fourier transform of q_N, implying that p_N has not the form of the generally used difference operator. Thus the momentum operator is approximated by the multiplication operator in the momentum representation

$$p_N|k\rangle = \frac{\hbar}{a}\eta_N k|k\rangle.$$

Schwinger's geometric idea was to identify Z_N with a *grid* in $\mathbb{R}$. For *N odd*, he defined a sequence of grids $L_N = \{a\eta_N\rho|\rho = 0, \pm 1, \ldots, \pm(N-1)/2\}$. In the limit $N \to \infty$ the grids are becoming denser and at the same time extending to the whole real line. The grid serves to embed the finite-dimensional Hilbert space $l^2(Z_N)$ isometrically in $L^2(\mathbb{R}, dq)$ by the map

$$\mathcal{I} : |\rho\rangle \mapsto \phi_\rho(q) = a^{-\frac{1}{2}}\eta_N^{-\frac{1}{2}}\chi_{[a\eta_N(\rho-\frac{1}{2}), a\eta_N(\rho+\frac{1}{2})]}(q),$$

where χ_S denotes the characteristic function of a subset $S \subset \mathbb{R}$. The position eigenvectors $|\rho\rangle$ are thus mapped onto narrow normalized wavefunctions $\phi_\rho(q)$ on the real line, centred at the grid points and contracting in the limit $N \to \infty$. Under the map $\mathcal{I}$ the normalizations of the wavefunctions $|\psi\rangle = \sum_\rho \psi_\rho|\rho\rangle$ and $(\mathcal{I}\psi)(q)$ are related by

$$|\psi_\rho|^2 = a\eta_N|(\mathcal{I}\psi)(a\eta_N\rho)|^2.$$

6. The finite-dimensional analogue of a quantum free particle

The finite-dimensional analogue of a quantum free particle was formulated in [10] as a discrete Galilean evolution along a finite closed linear chain. The single-step unitary time evolution operator C_N proposed there is diagonal in the momentum representation

$$\langle j|C_N|k\rangle = \delta_{jk}\omega_N^{-k^2}.$$

Transformation to the position representation gives

$$(C_N)_{\rho\sigma} = \sum_{jk}\langle \rho|j\rangle\langle j|C_N|k\rangle\langle k|\sigma\rangle = \frac{1}{N}\sum_{j=0}^{N-1}\omega_N^{-j^2+(\rho-\sigma)j}.$$

The unitary operator C_N fulfils the relations

$$C_N^{-1}Q_N C_N = \omega_N Q_N P_N^2, \qquad C_N^{-1}P_N C_N = P_N.$$

Looking at the free evolution in continuous phase space, we arrive at the conclusion that the operator C_N should be slightly modified. However, first consider the usual one-parameter group of unitary operators

$$T(t) = \exp\left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right), \qquad t \in \mathbb{R},$$

describing quantum evolution of a non-relativistic free particle of mass m on the real line. The corresponding $N \times N$ approximation is

$$T_N(\tau) = \exp\left(-\frac{i}{\hbar}\frac{p_N^2}{2m}\tau\varepsilon\right), \qquad \tau \in \mathbb{Z},$$

where we have introduced a time unit ε, since dynamically defined time intervals will play a special role. Thus t shall be restricted to integer multiples $\tau\varepsilon$, $\tau \in \mathbb{Z}$, of ε. The time unit ε will be chosen so that (in the momentum representation)

$$T_N(\tau)|j\rangle = \exp\left(-\frac{i}{\hbar}\frac{1}{2m}\left(\frac{\hbar}{a}\eta_N j\right)^2\tau\varepsilon\right)|j\rangle = \omega_N^{-\frac{1}{2}j^2\tau}|j\rangle, \qquad \tau \in \mathbb{Z},$$

including an additional $1/2$ factor in the exponent.[1] Our choice is in agreement with a dynamical relation

$$\varepsilon = \frac{ma^2}{\hbar} \qquad \text{or} \qquad m\frac{a}{\varepsilon} = \frac{\hbar}{a}$$

expressing the natural fact that a particle of momentum $\hbar/a$ traverses the distance a in time ε. Transformation to the position representation gives

$$T_N(\tau)_{\rho\sigma} = \sum_{jk}\langle \rho|j\rangle\langle j|T_N(\tau)|k\rangle\langle k|\sigma\rangle = \frac{1}{N}\sum_{j=0}^{N-1}\omega_N^{-\frac{1}{2}j^2\tau+(\rho-\sigma)j}.$$

Now in order to justify the factor of $1/2$ in the exponent recall that the free time evolution in continuous phase space $\mathbb{R}^2$ is described by translations along q with constant velocity,

$$q(t) = q(0) + \frac{p(0)}{m}t, \qquad p(t) = p(0).$$

The corresponding single-step 'translation' in quantum phase space Γ_N,

$$Q \mapsto QP, \qquad P \mapsto P$$

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[1] Non-integer powers of ω_N are understood as complex exponentials $\omega_N^w = \exp(\frac{2\pi i}{N}w)$, $w \in \mathbb{R}$.

is equivalent to the following $SL(2, \mathbb{Z}_N)$ transformation

$$(1, 1) = (1, 0) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad (0, 1) = (0, 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Its powers form an Abelian subgroup of $SL(2, \mathbb{Z}_N)$ isomorphic to $\mathbb{Z}_N$. An easy calculation shows that it is implemented by the unitary transformation

$$T_N(1)|j\rangle = \omega_N^{-\frac{1}{2}j^2}|j\rangle = C_{N1}|j\rangle,$$

which, from now on, will be denoted C_{N1}. The modified unitary operator C_{N1} now fulfils the relations

$$C_{N1}^{-1} Q_N C_{N1} = \omega_N^{\frac{1}{2}} Q_N P_N, \qquad C_{N1}^{-1} P_N C_{N1} = P_N, \qquad C_{N1}^{-s} Q_N^\rho P_N^j C_{N1}^s = \omega_N^{\frac{1}{2}\rho^2 s} Q_N^\rho P_N^{\rho s}.$$

7. $N \times N$ approximation of the Feynman path integral

Let $|q(0), 0\rangle$ and $|q(t), t\rangle$ be the state vectors of the initial state and of the final state of a particle on $\mathbb{R}$ at times $0, t$, respectively. If $S[q]$ is the classical action functional of the particle, the evolution amplitude is according to Feynman formally written as a path integral [2]

$$\langle q(t), t | q(0), 0 \rangle = \int e^{\frac{i}{\hbar} S[q]} \mathcal{D}q(t).$$

It is understood as a sum over all continuous paths in configuration space. According to Feynman's principle of equivalence of trajectories, the contribution of each path should have the same absolute value, and hence contributes to the sum only a phase factor with the phase given by the classical action in units $\hbar$ evaluated along the path.

In quantum mechanics, the path integral is traditionally defined as a limit via discretization based on the division of the time interval, e.g. into n intervals of equal duration $\varepsilon = t/n$. The evolution amplitude is thus written as a multiple integral [2]

$$\langle q(t), t | q(0), 0 \rangle = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \langle q(t)|e^{-\frac{i}{\hbar}H\varepsilon}|q_{n-1}\rangle \, \mathrm{d}q_{n-1} \ldots \mathrm{d}q_1 \langle q_1|e^{-\frac{i}{\hbar}H\varepsilon}|q(0)\rangle,$$

where $q_l = q(l\varepsilon)$ and H is the Hamilton operator. Each factor—the short-time propagator—is then identified with an exponential of the short-time action involving an approximation of the classical Lagrangian,

$$\langle q_{l+1}|e^{-\frac{i}{\hbar}H\varepsilon}|q_l\rangle = \frac{1}{A} e^{\frac{i}{\hbar}L(q_{l+1},q_l)\varepsilon},$$

with normalization factor A. For instance, for a non-relativistic particle of mass m

$$H = \frac{\hat{p}^2}{2m} + V(\hat{q}),$$

and one computes (via the momentum representation)

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}\left(p_l \frac{q_{l+1} - q_l}{\varepsilon} - \frac{p_l^2}{2m} - V(q_l)\right)\varepsilon\right) \mathrm{d}p_l$$

$$= \left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{-\frac{1}{2}} \exp\left(\frac{i}{\hbar}\left(\frac{1}{2}m\left(\frac{q_{l+1} - q_l}{\varepsilon}\right)^2 - V(q_l)\right)\varepsilon\right), \tag{2}$$

i.e.

$$L(q_{l+1}, q_l) = \frac{1}{2}m\left(\frac{q_{l+1} - q_l}{\varepsilon}\right)^2 - V(q_l) \qquad \text{and} \qquad A = \left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{\frac{1}{2}}.$$

A sequence of q_l's for each t_l shall, in the limit, define a path of the system and each of the integrals is to be taken over the entire range available to each q_l. In other words, the multiple integral is taken over all possible paths.[2]

Let us return to our analogue of a *free non-relativistic particle*. The above approach will guide us in our $N \times N$ approximation with the short-time propagator induced by the unitary operator C_{N1}. In this approximation $q_l \approx a\eta_N \rho_l$, so, for a single time step, $\langle q_{l+1}, \varepsilon | q_l, 0 \rangle$ is approximated by

$$\langle q_{l+1}, \varepsilon | q_l, 0 \rangle a\eta_N = \langle q_{l+1} | e^{-\frac{i}{\hbar} H \varepsilon} | q_l \rangle a\eta_N = \langle \rho_{l+1} | C_{N1} | \rho_l \rangle = \frac{1}{N} \sum_{j_l=0}^{N-1} \omega_N^{-\frac{1}{2} j_l^2 + (\rho_{l+1} - \rho_l) j_l}.$$

For τ time steps

$$\langle \rho_\tau, \tau\varepsilon | \rho_0, 0 \rangle = \sum_{\rho_1, \ldots, \rho_{\tau-1}} \langle \rho_\tau | C_{N1} | \rho_{\tau-1} \rangle \ldots \langle \rho_1 | C_{N1} | \rho_0 \rangle = \langle \rho_\tau | C_{N1}^\tau | \rho_0 \rangle$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{-\frac{1}{2} j^2 \tau + (\rho_\tau - \rho_0) j}.$$

The above Gauss-like sum for a single time step can be summed up using Siegel's reciprocity formula for generalized Gauss sums [17–19]

$$\sum_{n=0}^{|c|-1} e^{\pi i (an^2 + bn)/c} = \sqrt{\left| \frac{c}{a} \right|} e^{\pi i (|ac| - b^2)/(4ac)} \sum_{n=0}^{|a|-1} e^{-\pi i (cn^2 + bn)/a},$$

valid for $a, b, c \in \mathbb{Z}$, $ac \neq 0$, $ac + b$ even. Putting $a = N$ with N *odd*, $c = 1$, $n = j_l$ and $b = -2\rho - 1$ with $\rho = \rho_{l+1} - \rho_l$ one obtains

$$\frac{1}{N} \sum_{j_l=0}^{N-1} \omega_N^{-\frac{1}{2} j_l(j_l-1) + (\rho_{l+1} - \rho_l) j_l} = \frac{1}{\sqrt{iN}} \omega_N^{\frac{1}{2}(\rho_{l+1} - \rho_l + \frac{1}{2})^2}.$$

On the basis of this formula we prefer the operator

$$C_{N2} | j \rangle = \omega_N^{-\frac{1}{2} j(j-1)} | j \rangle$$

for unitary single-step time evolution. It satisfies simpler relations than C_{N1},

$$C_{N2}^{-1} Q_N C_{N2} = Q_N P_N, \qquad C_{N2}^{-1} P_N C_{N2} = P_N, \qquad C_{N2}^{-s} Q_N^\rho P_N^j C_{N2}^s = Q_N^\rho P_N^{\rho s},$$

while inducing the same Abelian subgroup of translations of quantum phase space. With operator C_{N2} we compute in the position representation

$$\langle \rho_{l+1} | C_{N2} | \rho_l \rangle = \frac{1}{\sqrt{iN}} \omega_N^{\frac{1}{2}(\rho_{l+1} - \rho_l + \frac{1}{2})^2}.$$

This result can be interpreted as the *emergence of a dimensionless Lagrangian* $\mathcal{L}_N$,

$$\langle \rho_{l+1} | C_{N2} | \rho_l \rangle = \frac{1}{\sqrt{iN}} \omega_N^{\mathcal{L}_N(\rho_{l+1}, \rho_l)}, \qquad \mathcal{L}_N(\rho_{l+1}, \rho_l) = \frac{1}{2} \left(\rho_{l+1} - \rho_l + \frac{1}{2} \right)^2.$$

In order to go over to the $1 + 1$ space time and obtain the corresponding local Lagrangian L_N, we divide by $a\eta_N$ and express the short-time propagator

$$\langle q_{l+1}, \varepsilon | q_l, 0 \rangle = \frac{1}{a\eta_N} \langle \rho_{l+1} | C_{N2} | \rho_l \rangle = \frac{1}{a\eta_N} \frac{1}{\sqrt{iN}} \omega_N^{\frac{1}{2}(\rho_{l+1} - \rho_l + \frac{1}{2})^2} = \left(\frac{2\pi i \hbar \varepsilon}{m} \right)^{-\frac{1}{2}} e^{\frac{i}{\hbar} \frac{1}{2} m \left(\frac{q_{l+1} - q_l + \frac{a\eta_N}{2}}{\varepsilon} \right)^2 \varepsilon}.$$

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[2] Recall a quotation from Feynman's thesis (p 69 of [1]): *'A point of vagueness is the normalization factor, A. No rule has been given to determine it for a given action expression. This question is related to the difficult mathematical question as to the conditions under which the limiting process of subdividing the time scale, required by equations such as (68), actually converges.'*

This result can be rewritten as

$$\frac{1}{\sqrt{iN}}\omega_N^{\mathcal{L}_N(\rho_{l+1},\rho_l)} = \left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{-\frac{1}{2}} e^{\frac{i}{\hbar}L_N(q_{l+1},q_l)\varepsilon} a\eta_N,$$

where the phase factor appearing in the short-time propagator is seen to define the corresponding small increment of the action $L_N\varepsilon$ which is proportional to the local Lagrangian

$$L_N = \frac{1}{2}m\left(\frac{q_{l+1} - q_l + \frac{a\eta_N}{2}}{\varepsilon}\right)^2.$$

Note that in the limit $N \to \infty$, we have $\eta_N = \sqrt{2\pi/N} \to 0$ and obtain the usual form of the short-time propagator for the free quantum particle.

We close this section with a one-dimensional particle moving in a potential field $V(q)$. This potential has been incorporated in the short-time propagator (2). To get its $N \times N$ approximation, the potential $V(q)$ is sampled only at the grid points $q_l = a\eta_N\rho_l$, $\rho_l = -(N-1)/2, \ldots, (N-1)/2$. In order to transform $V(q_l)$ into a dimensionless form it should be expressed in the energy unit

$$\frac{1}{m}\left(\eta_N\frac{\hbar}{a}\right)^2 = \frac{2\pi}{N}\frac{\hbar}{\varepsilon}$$

used for transforming the kinetic energy to $j^2/2$,

$$V(q_l) = V(a\eta_N\rho_l) = \frac{2\pi}{N}\frac{\hbar}{\varepsilon}w_l.$$

As a result, the potential is represented by a set of N dimensionless constants w_l. Thus the short-time propagator obtained for a free particle is subject only to a slight modification by constants w_l:

$$\langle q_{l+1}, \varepsilon | q_l, 0\rangle = \frac{1}{a\eta_N}\langle\rho_{l+1}|C_{N2}\omega_N^{-w_l}|\rho_l\rangle = \frac{1}{a\eta_N}\frac{1}{\sqrt{iN}}\omega_N^{\frac{1}{2}(\rho_{l+1}-\rho_l+\frac{1}{2})^2 - w_l}$$

$$= \left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{-\frac{1}{2}}\exp\left(\frac{i}{\hbar}\left[\frac{1}{2}m\left(\frac{q_{l+1} - q_l + \frac{a\eta_N}{2}}{\varepsilon}\right)^2 - V(q_l)\right]\varepsilon\right).$$

Also in these formulae the *emergence of a local Lagrangian* is clearly manifest.

8. The short-time propagator and mutually unbiased bases

Let us denote the bases composed of eigenvectors of the operators $Q_N^j P_N^\sigma$ by $\mathcal{B}_{(j,\sigma)}$. The unitary operator C_{N2} (or C_{N1}) plays an analogous role to operator D_N in our previous study of mutually unbiased bases *for prime N* [20]. There the iterations of D_N generated the *maximal set of $N + 1$ mutually unbiased bases*

$$\mathcal{B}_{(1,0)} \overset{S_N}{\to} \mathcal{B}_{(0,1)} \overset{D_N}{\to} \mathcal{B}_{(1,1)} \overset{D_N}{\to} \mathcal{B}_{(2,1)} \overset{D_N}{\to} \ldots \overset{D_N}{\to} \mathcal{B}_{(N-1,1)},$$

starting with the canonical basis $\mathcal{B}_{(1,0)}$.

If N is prime, identical reasoning to that in [20] shows that the iterations of the unitary operator C_N (indices 1 or 2 omitted) generate in a similar way another maximal set of $N + 1$ mutually unbiased bases

$$\mathcal{B}_{(0,1)} \overset{S_N^{-1}}{\to} \mathcal{B}_{(1,0)} \overset{C_N}{\to} \mathcal{B}_{(1,1)} \overset{C_N}{\to} \mathcal{B}_{(1,2)} \overset{C_N}{\to} \ldots \overset{C_N}{\to} \mathcal{B}_{(1,N-1)},$$

now starting with the momentum basis $\mathcal{B}_{(0,1)}$.   Thus the composite unitary operators $C_N^b S_N^{-1}$, $b = 0, 1, \ldots, N-1$ produce all the bases of the maximal set when applied to the momentum basis.

In this paper N is not restricted to primes but may take an arbitrary odd value. Notwithstanding this general situation the formulae derived in the previous section show that the bases appearing in the short-time propagators, i.e. $\{|\rho\rangle\} = \mathcal{B}_{(1,0)}$ and $\left\{C_N \omega_N^{-w_l}|\rho\rangle\right\} = \mathcal{B}_{(1,1)}$, are mutually unbiased. Especially, for the short-time propagator

$$\left|\langle \rho_{l+1}|C_N \omega_N^{-w_l}|\rho_l\rangle\right| = \frac{1}{\sqrt{N}} \tag{3}$$

holds and it is this constant absolute value that entails mutual unbiasedness of the bases involved. From the physical viewpoint *the state evolving after a short time interval ε carries no information about the preceding state*. The trivial information-theoretic meaning of mutual unbiasedness therefore consists in the fact that in each single evolution step complete loss of information occurs. Physical information is carried only by the phase factor whose phase is proportional to the local Lagrangian.

9. Conclusions

The foregoing development of $N \times N$ approximations of the Feynman path integral is a continuation of our previous studies of quantum mechanics over finite configuration spaces [10, 20] and of discrete path summation [21]. Our approximation method was first applied to discrete time evolution of an analogue of a quantum free non-relativistic particle in discrete 1+1 space time. The powers of the evolution operator can be interpreted as a unitary representation in $l^2(\mathbb{Z}_\mathbb{N})$ of an Abelian subgroup of $SL(2, \mathbb{Z}_\mathbb{N})$ acting on a quantum phase space $\mathbb{Z}_\mathbb{N} \times \mathbb{Z}_\mathbb{N}$. Generalization of the $N \times N$ approximation of the short-time propagator to a particle moving in a potential turned out to be straightforward.

Summarizing, we have shown explicitly how a Lagrangian arises as a local phase in the short-time propagator in the $N \times N$ approximation, thus confirming Svetlichny's conjecture. Our paper also brings definite answers to the questions quoted in the introduction in the case of a *one-dimensional non-relativistic particle moving in a potential field*.

(i) Our results concern unitary groups $U(t)$ in $L^2(\mathbb{R})$ connected with the time evolution of a one-dimensional non-relativistic quantum particle moving in a potential field. Equation (3) shows that position bases involved in the $N \times N$ approximation of the short-time propagator for arbitrary odd N are mutually unbiased. In this sense also in the limit $N \to \infty$ position bases at times 0 and t do tend to mutual unbiasedness as $t \to 0$.

(ii) Our discrete analogue of quantum evolution of a free particle in a finite-dimensional Hilbert space which accurately simulates the Galilean evolution of a free particle on the real line was consistently employed throughout the paper.

(iii) In our paper also the secret of the prefactor A^{-1} in the short-time propagator is unveiled. Its dimension must be inverse length in order to compensate the integration over q in the Feynman path integral. Hence in the $N \times N$ approximation it must be related to our unit of length $a\eta_N$. Due to $\varepsilon = ma^2/\hbar$,

$$\frac{a\eta_N}{A} = a\sqrt{\frac{2\pi}{N}}\sqrt{\frac{m}{2\pi i\hbar\varepsilon}} = \frac{1}{\sqrt{iN}}$$

holds for single-step time evolution. This relation shows the trivial information-theoretic meaning of A^{-1}: *in the dimensionless expression it is a constant corresponding to complete loss of information in each single time step*.

In connection with the $N \times N$ approximation we would like to point out the suggestion of the approximate solution of the continuous Schrödinger equation. Namely, Digernes, Husstad and Varadarajan [22] proved a convergence theorem on approximation of continuous Weyl systems by $N \times N$ Weyl operators $Q_N^j P_N^\sigma$. Further, Digernes, Varadarajan and Varadhan [23] proved a strong theorem on convergence of eigenvalues and eigenfunctions of $N \times N$ Hamiltonians to solutions of the one-dimensional Schrödinger equation for potentials satisfying $V \to +\infty$ as $|q| \to \infty$, hence possessing a discrete spectrum. This theorem provides a justification for the approximate solution of the continuous Schrödinger equation. Numerical calculations showed that the approximation is unexpectedly good even for relatively small values of N. The generalization of these results in the case of a mixed spectrum remains open. Let us note that in quantum optics, a discrete phase space $\mathbb{Z}_n \times \mathbb{Z}_n$ is employed in the discrete approximation of the quantum phase and the conjugate number operator [24].

Acknowledgments

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